Radbound University

A Coq proof of the correctness of X25519 in TweetNaCl

Peter Schwabe, **Benoît Viguier**, Timmy Weerwag, Freek Wiedijk

Crypto Working Group
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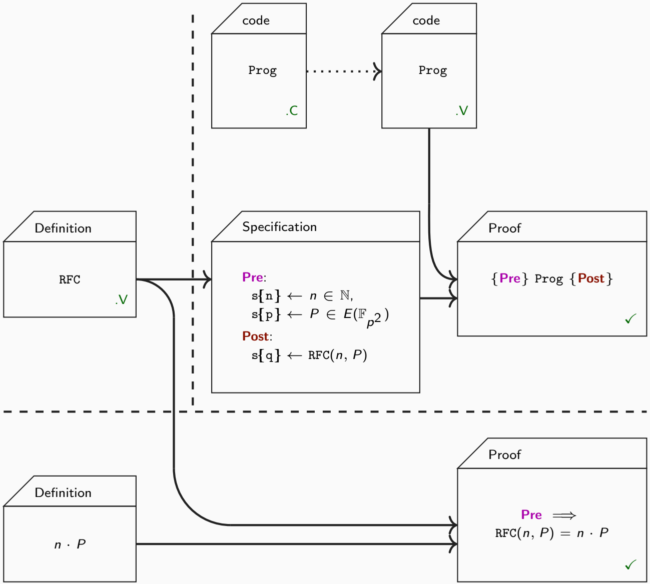
Institute for Computing and Information Sciences – Digital Security
Radboud University, Nijmegen

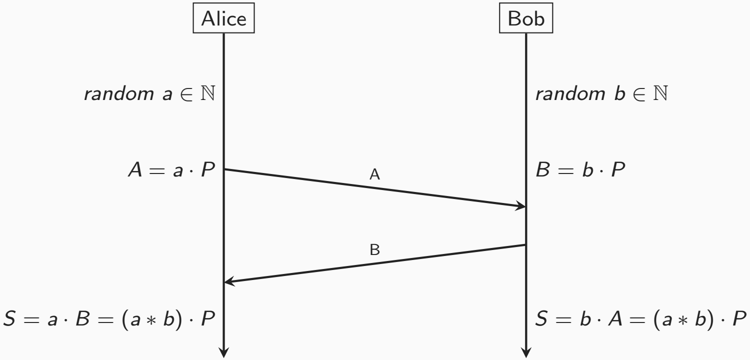
Overview


code
Prog
.C
code
Prog
.V
Definition
RFC
.V
Specification
Pre:
s[n] ← $n \in \mathbb{N}$,
s[p] ← $P \in E(\mathbb{F}_{p^2})$
Post:
s[q] ← RFC(n, P)
Proof
{Pre} Prog {Post}
✓
Definition
$n \cdot P$
Proof
Pre ⟹
RFC(n, P) = $n \cdot P$
✓

Prelude

Diffie-Hellman with Elliptic Curves

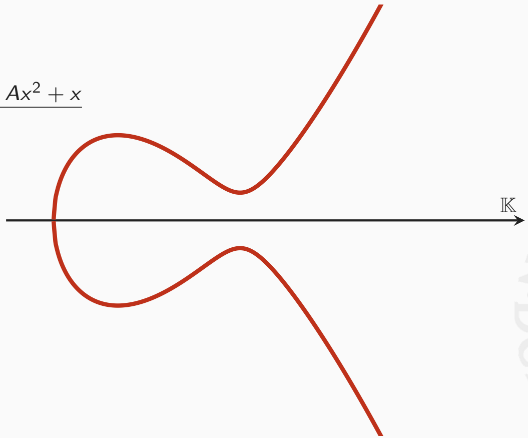
Public parameter: point P, curve E over $\mathbb{K}$



<u>*Operations on* $E : By^2 = x^3 + Ax^2 + x$</u>

(1) $P \mapsto [2]P$

(2) $\{P, Q\} \mapsto P + Q$



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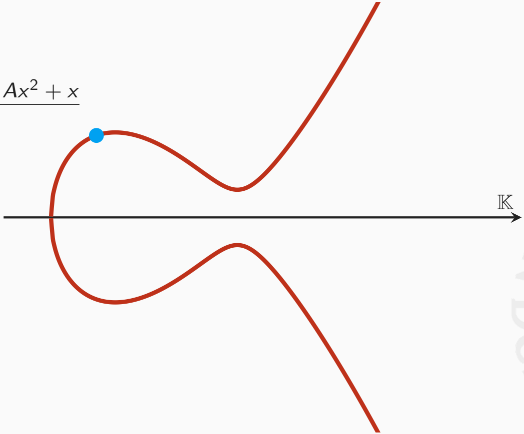

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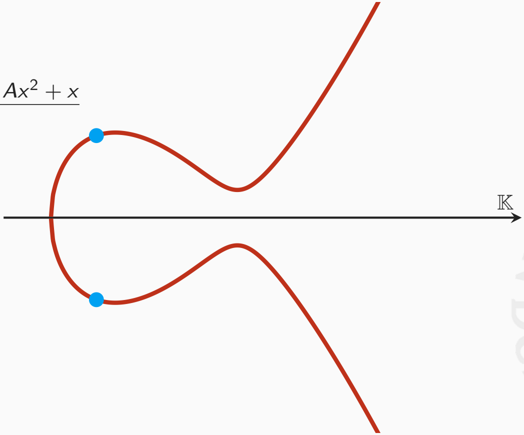

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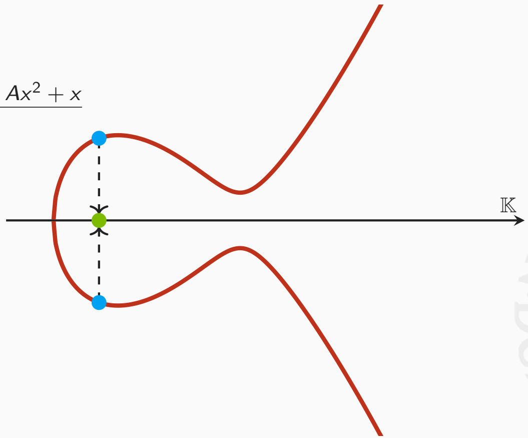

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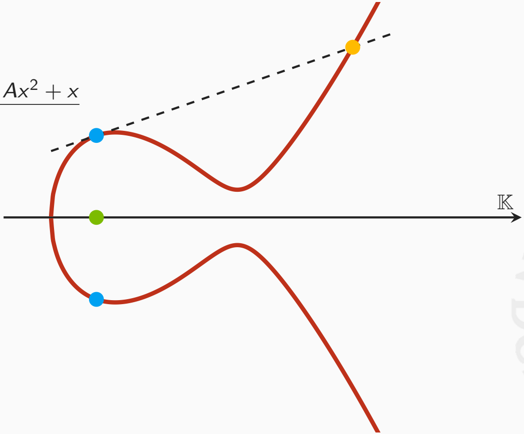

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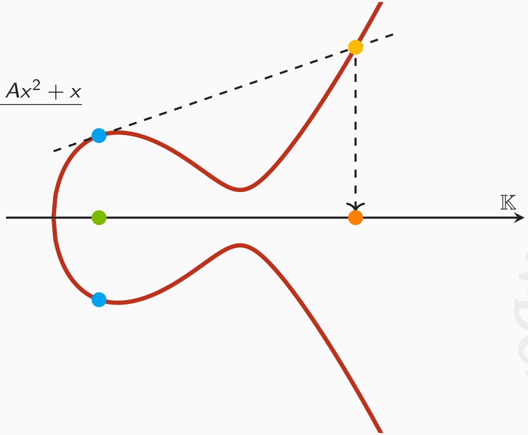

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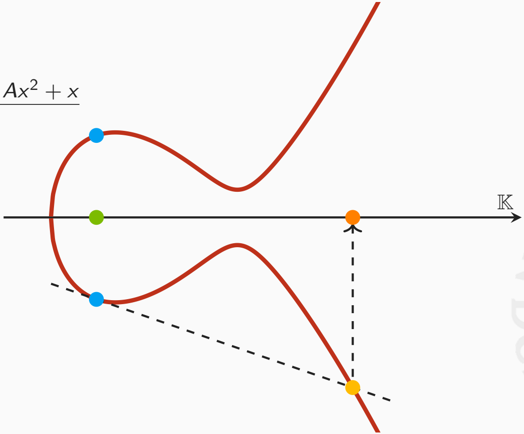

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Operations on $\mathbb{P}$

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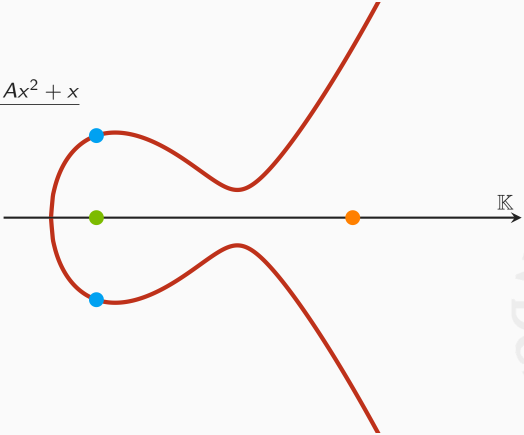

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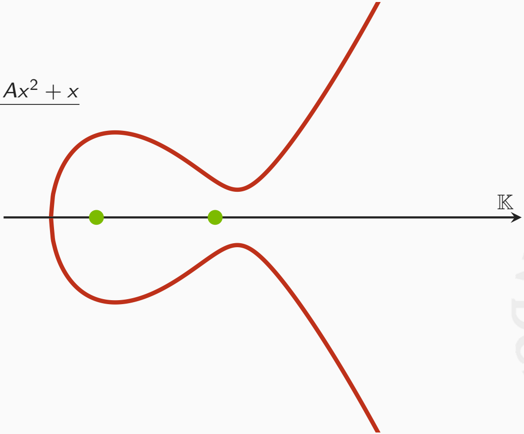

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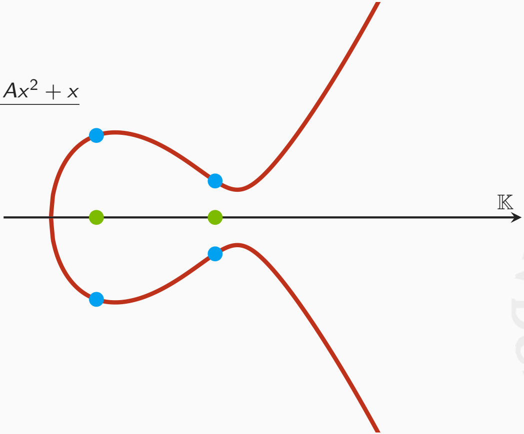

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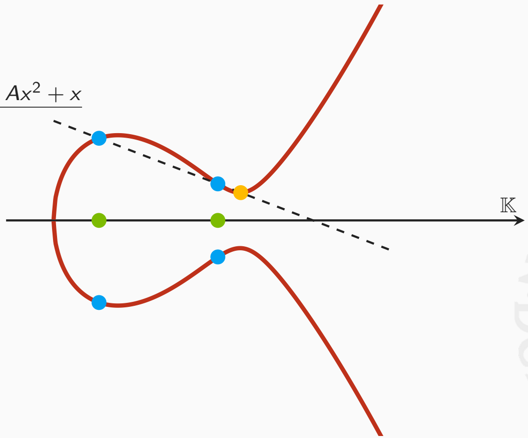

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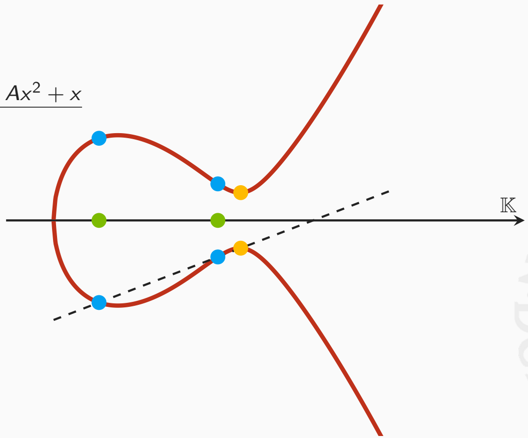

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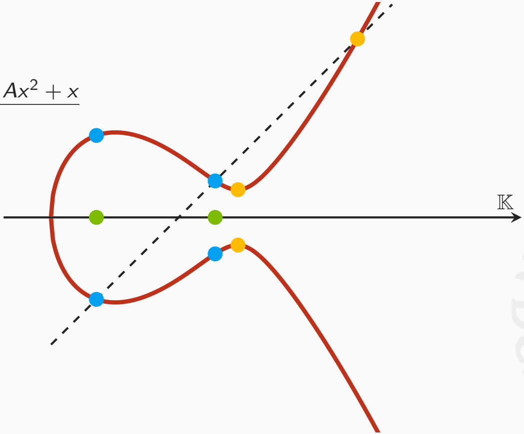

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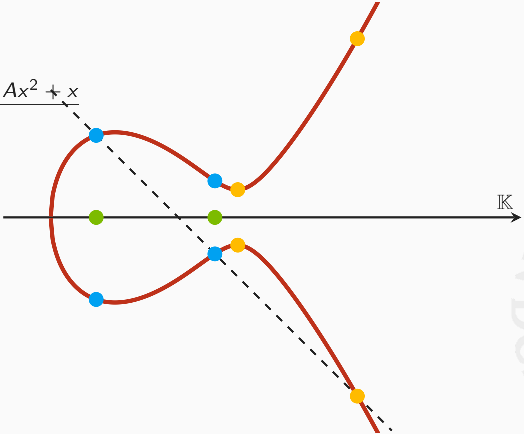

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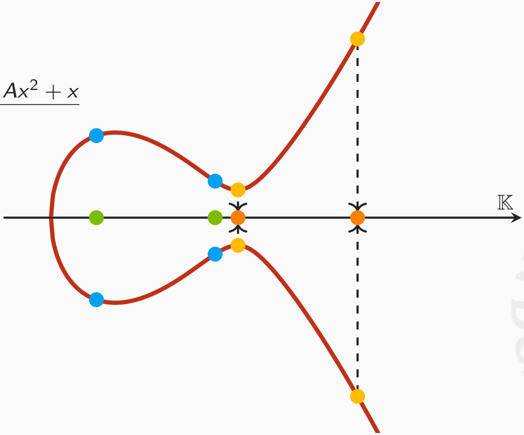

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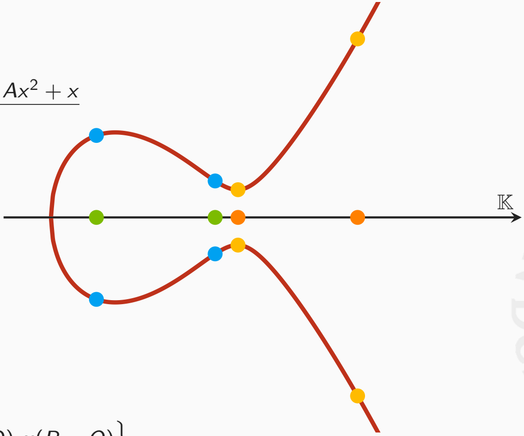

Operations on $\mathbb{P}$

(1) $\texttt{xDBL} : \mathbf{x}(P) \mapsto \mathbf{x}([2]P)$

(2) $\left\{\mathbf{x}(P), \mathbf{x}(Q)\right\} \mapsto \left\{\mathbf{x}(P+Q), \mathbf{x}(P-Q)\right\}$

Operations on $E : By^2 = x^3 + Ax^2 + x$

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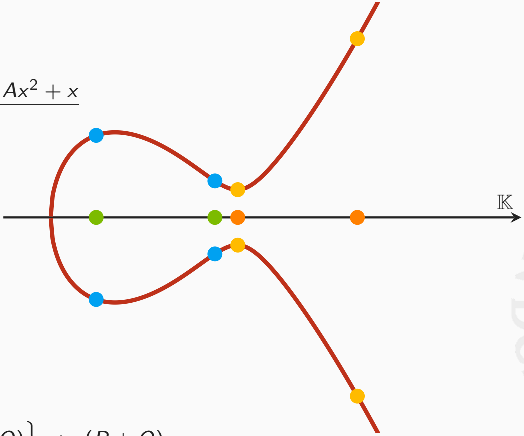


Operations on $\mathbb{P}$

(1) $\texttt{xDBL} : \mathbf{x}(P) \mapsto \mathbf{x}([2]P)$

(2) $\texttt{xADD} : \left\{\mathbf{x}(P), \mathbf{x}(Q), \mathbf{x}(P-Q)\right\} \mapsto \mathbf{x}(P+Q)$

Montgomery ladder

Algorithm 1 Montgomery ladder for scalar mult.

Input: x-coordinate x_P of a point P, scalar n with $n < 2^m$

Output: x-coordinate x_Q of $Q = n \cdot P$

$Q = (X_Q : Z_Q) \leftarrow (1 : 0)$
$R = (X_R : Z_R) \leftarrow (x_P : 1)$
for $k := m$ down to 1 **do**
  $(Q, R) \leftarrow \texttt{CSWAP}((Q, R), k^{\text{th}}$ bit of $n)$
  $Q \leftarrow \texttt{xDBL}(Q)$
  $R \leftarrow \texttt{xADD}(x_P, Q, R)$
  $(Q, R) \leftarrow \texttt{CSWAP}((Q, R), k^{\text{th}}$ bit of $n)$
end for
return X_Q / Z_Q

A quick overview of TweetNaCl

```
int crypto_scalarmult(u8 *q,const u8 *n,const u8 *p)
{
  u8 z[32]; i64 r; int i; gf x,a,b,c,d,e,f;
  FOR(i,31) z[i]=n[i];
  z[31]=(n[31]&127)|64; z[0]&=248;               #   Clamping of n
  unpack25519(x,p);
  FOR(i,16) { b[i]=x[i]; d[i]=a[i]=c[i]=0; }
  a[0]=d[0]=1;
  for(i=254;i>=0;--i) {
    r=(z[i>>3]>>(i&7))&1;                        #   i^th bit of n
    sel25519(a,b,r);
    sel25519(c,d,r);
    A(e,a,c);                                    #
    Z(a,a,c);                                    #
    A(c,b,d);                                    #
    Z(b,b,d);                                    #
    S(d,e);                                      #
    S(f,a);                                      #
    M(a,c,a);                                    #  Montgomery Ladder
    M(c,b,e);                                    #
    A(e,a,c);                                    #
    Z(a,a,c);                                    #
    S(b,a);                                      #
    Z(c,d,f);                                    #
    M(a,c,_121665);                              #
    A(a,a,d);                                    #
    M(c,c,a);                                    #
    M(a,d,f);                                    #
    M(d,b,x);                                    #
    S(b,e);                                      #
    sel25519(a,b,r);
    sel25519(c,d,r);
  }
  inv25519(c,c); M(a,a,c);                       #   a / c
  pack25519(q,a);
  return 0;
```

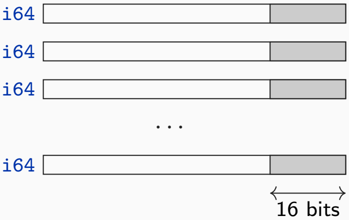
Number representation

256-bits integers do not fit into a 64-bits containers...

256 bits number

16×16 bits limbs

```c
typedef long long i64;
typedef i64 gf[16];
```



Basic Operations

```
#define FOR(i,n) for (i = 0;i < n;++i)
#define sv static void
typedef long long i64;
typedef i64 gf[16];


sv A(gf o,const gf a,const gf b)    # Addition
{
  int i;
  FOR(i,16) o[i]=a[i]+b[i];         # carrying is done separately
}


sv Z(gf o,const gf a,const gf b)    # Zubtraction
{
  int i;
  FOR(i,16) o[i]=a[i]-b[i];         # carrying is done separately
}


sv M(gf o,const gf a,const gf b)    # Multiplication (school book)
{
  i64 i,j,t[31];
  FOR(i,31) t[i]=0;
  FOR(i,16) FOR(j,16) t[i+j] = a[i]*b[j];
  FOR(i,15) t[i]+=38*t[i+16];
  FOR(i,16) o[i]=t[i];
  car25519(o);                      # carrying
  car25519(o);                      # carrying
}
```

Formalizing X25519 from RFC 7748

The specification of X25519 in RFC 7748 is formalized by `RFC` in Coq.

More formally:

```
Definition RFC (n: list Z) (p: list Z) : list Z :=
  let k := decodeScalar25519 n in
  let u := decodeUCoordinate p in
  let t := montgomery_rec
    255  (* iterate 255 times *)
    k    (* clamped n         *)
    1    (* x2                *)
    u    (* x3                *)
    0    (* z2                *)
    1    (* z3                *)
    0    (* dummy             *)
    0    (* dummy             *)
    u    (* x1                *) in
  let a := get_a t in
  let c := get_c t in
  let o := ZPack25519 (Z.mul a (ZInv25519 c))
  in encodeUCoordinate o.
```

```
Fixpoint montgomery_rec (m : nat) (z : T')
(a: T) (b: T) (c: T) (d: T) (e: T) (f: T) (x: T) :
(* a: x2    b: x3    c: z2    d: z3    x: x1   *)
(T * T * T * T * T * T) :=
match m with
| 0%nat ⇒  (a,b,c,d,e,f)
| S n ⇒
  let r := Getbit (Z.of_nat n) z in                    (* k_t = (k >> t) & 1                 *)
    (* swap ←  k_t *)
  let (a, b) := (Sel25519 r a b, Sel25519 r b a) in (* (x2, x3) = cswap(swap, x2, x3)   *)
  let (c, d) := (Sel25519 r c d, Sel25519 r d c) in (* (z2, z3) = cswap(swap, z2, z3)   *)
  let e := a + c in                                   (* A = x2 + z2                        *)
  let a := a - c in                                   (* B = x2 - z2                        *)
  let c := b + d in                                   (* C = x3 + z3                        *)
  let b := b - d in                                   (* D = x3 - z3                        *)
  let d := e ^2 in                                    (* AA = A^2                           *)
  let f := a ^2 in                                    (* BB = B^2                           *)
  let a := c * a in                                   (* CB = C * B                         *)
  let c := b * e in                                   (* DA = D * A                         *)
  let e := a + c in                                   (* x3 = (DA + CB)^2                   *)
  let a := a - c in                                   (* z3 = x1 * (DA - CB)^2              *)
  let b := a ^2 in                                    (* z3 = x1 * (DA - CB)^2              *)
  let c := d - f in                                   (* E = AA - BB                        *)
  let a := c * C_121665 in                            (* z2 = E * (AA + a24 * E)            *)
  let a := a + d in                                   (* z2 = E * (AA + a24 * E)            *)
  let c := c * a in                                   (* z2 = E * (AA + a24 * E)            *)
  let a := d * f in                                   (* x2 = AA * BB                       *)
  let d := b * x in                                   (* z3 = x1 * (DA - CB)^2              *)
  let b := e ^2 in                                    (* x3 = (DA + CB)^2                   *)
  let (a, b) := (Sel25519 r a b, Sel25519 r b a) in (* (x2, x3) = cswap(swap, x2, x3)   *)
  let (c, d) := (Sel25519 r c d, Sel25519 r d c) in (* (z2, z3) = cswap(swap, z2, z3)   *)
  montgomery_rec n z a b c d e f x
end.
```

RFC 7748 in Coq

Let $\texttt{ZofList} : \mathbb{Z} \to \texttt{list}\ \mathbb{Z} \to \mathbb{Z}$*, a function given n and a list l returns its little endian decoding with radix* 2^n*.*

```
Fixpoint ZofList {n:Z} (a:list Z) : Z :=
  match a with
  | [] => 0
  | h :: q => h + 2^n * ZofList q
  end.
```

Let $\texttt{ListofZ32} : \mathbb{Z} \to \mathbb{Z} \to \texttt{list}\ \mathbb{Z}$*, given n and a returns a's little-endian encoding as a list with radix* 2^n*.*

```
Fixpoint ListofZn_fp {n:Z} (a:Z) (f:nat) : list Z :=
match f with
  | 0%nat => []
  | S fuel => (a mod 2^n) :: ListofZn_fp (a/2^n) fuel
end.

Definition ListofZ32 {n:Z} (a:Z) : list Z :=
  ListofZn_fp n a 32.
```

ListofZ32 and ZofList are inverse to each other.

```
Lemma ListofZ32_ZofList_Zlength: forall (l:list Z),
  Forall (λx ⇒ 0 ≤ x < 2^n) l →
  Zlength l = 32 →
  ListofZ32 n (ZofList n l) = l.
Qed.
```

With those tools at hand, we formally define the decoding and encoding as specified in the RFC.

```
Definition decodeScalar25519 (l: list Z) : Z :=
  ZofList 8 (clamp l).

Definition decodeUCoordinate (l: list Z) : Z :=
  ZofList 8 (upd_nth 31 l
    (Z.land (nth 31 l 0) 127)).

Definition encodeUCoordinate (x: Z) : list Z :=
  ListofZ32 8 x.
```

From C to Coq

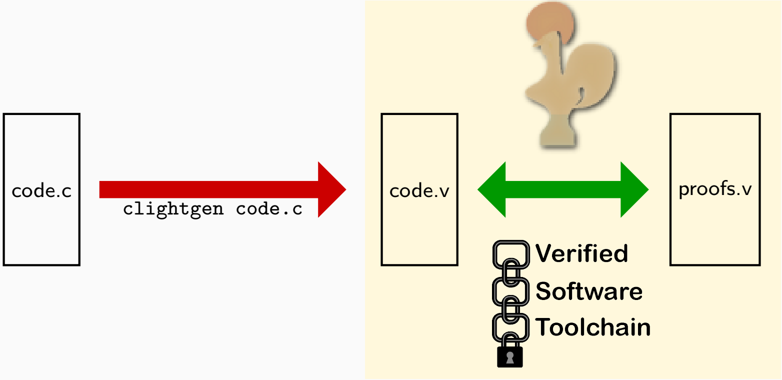
$$\{\textbf{Pre}\}\ \texttt{Prog}\ \{\textbf{Post}\}$$

where **Pre** and **Post** are assertions and `Prog` is a fragment of code.

"when the precondition **Pre** is met, executing `Prog` will yield postcondition **Post**".

Sequent Rule in Hoare logic:

$$\text{Hoare-Seq}\ \frac{\{P\}C_1\{Q\} \qquad \{Q\}C_2\{R\}}{\{P\}C_1;C_2\{R\}}$$


code.c
clightgen code.c
code.v
proofs.v
Verified
Software
Toolchain

Specification: Addition

```
Fixpoint Low.A (a b : list ℤ) : list ℤ :=
  match a,b with
  | [], q ⇒ q
  | q,[] ⇒ q
  | h1::q1,h2::q2 ⇒ (Z.add h1 h2) :: Low.A q1 q2
  end.
Notation "a ⊞ b" := (Low.A a b) (at level 60).


Corollary A_correct:
  forall (a b: list ℤ),
    ZofList 16 (a ⊞ b) = (ZofList 16 a) + (ZofList 16 b).
Qed.


Lemma A_bound_len:
  forall (m1 n1 m2 n2: ℤ) (a b: list ℤ),
    length a = length b →
    Forall (λx ⇒ m1 < x < n1) a →
    Forall (λx ⇒ m2 < x < n2) b →
      Forall (λx ⇒ m1 + m2 < x < n1 + n2) (a ⊞ b).
Qed.

Lemma A_length_16:
  forall (a b: list ℤ),
  length a = 16 →
  length b = 16 →
    length (a ⊞ b) = 16.
Qed.
```

```
Definition A_spec :=
DECLARE _A
WITH
  v_o: val, v_a: val, v_b: val,
  sh : share,
  o : list val,
  a : list Z, amin : Z, amax : Z,
  b : list Z, bmin : Z, bmax : Z,

(*----------------------------------------*)
PRE [ _o OF (tptr tlg), _a OF (tptr tlg), _b OF (tptr tlg) ]
    PROP  (writable_share sh;
          (* For soundness *)                          (* For bounds propagation *)
           Forall (λx ↦ -2^62 < x < 2^62) a;           Forall (λx ↦ amin < x < amax) a;
           Forall (λx ↦ -2^62 < x < 2^62) b;           Forall (λx ↦ bmin < x < bmax) b;

           Zlength a = 16; Zlength b = 16; Zlength o = 16)
    LOCAL (temp _a v_a; temp _b v_b; temp _o v_o)
    SEP   (sh{ v_o }←(lg16)− o;
           sh{ v_a }←(lg16)− mVI64 a;
           sh{ v_b }←(lg16)− mVI64 b)

  (*----------------------------------------*)
  POST [ tvoid ]
      PROP ( (* Bounds propagation *)
            Forall (λx ↦ amin + bmin < x < amax + bmax) (Low.A a b)
            Zlength (A a b) = 16;
           )
      LOCAL()
      SEP (sh{ v_o }←(lg16)− mVI64 (Low.A a b);
           sh{ v_a }←(lg16)− mVI64 a;
           sh{ v_b }←(lg16)− mVI64 b).
```

```
sv A(gf o,const gf a,const gf b)
{
  int i;
  FOR(i,16) o[i]=a[i]+b[i];
}
```

Crypto_Scalarmult and RFC

(1) We define `Low.A`; `Low.M`; `Low.Sq`; `Low.Zub`; `Unpack25519`; `clamp`; `Pack25519`; `Inv25519`; `car25519` to have the same behavior as the low level C code.

(2) We define `Crypto_Scalarmult` with `Low.A`; `Low.M`; `Low.Sq`; `Low.Zub`; `Unpack25519`; `clamp`; `Pack25519`; `Inv25519`; `car25519`; `montgomery_rec`.

(3) We prove that `Low.M`; `Low.A`; `Low.Sq`; `Low.Zub`; `Unpack25519`; `clamp`; `Pack25519`; `Inv25519`; `car25519` have the same behavior over `list Z` as their equivalent over `Z` with $:GF$ (in $\mathbb{Z}_{2^{255}-19}$).

(4) We prove that `Crypto_Scalarmult` performs the same computation as `RFC`.

```
Lemma Crypto_Scalarmult_RFC_eq :
  forall (n: list Z) (p: list Z),
  Zlength n = 32 →
  Zlength p = 32 →
  Forall (λ x ⇒ 0 ≤ x ∧ x < 2^8) n →
  Forall (λ x ⇒ 0 ≤ x ∧ x < 2^8) p →
  Crypto_Scalarmult n p = RFC n p.
Qed.
```

Hoare Triple of crypto_scalarmult

```
Definition crypto_scalarmult_spec :=
DECLARE _crypto_scalarmult_curve25519_tweet
WITH
  v_q: val, v_n: val, v_p: val, c121665:val,
  sh : share,
  q : list val, n : list Z, p : list Z
(*----------------------------------------*)
PRE [ _q OF (tptr tuchar), _n OF (tptr tuchar), _p OF (tptr tuchar) ]
PROP (writable_share sh;
      Forall (λx ↦ 0 ≤ x < 2^8) p;
      Forall (λx ↦ 0 ≤ x < 2^8) n;
      Zlength q = 32;
      Zlength n = 32;
      Zlength p = 32)
LOCAL(temp _q v_q; temp _n v_n; temp _p v_p; gvar __121665 c121665)
SEP  (sh{ v_q }←(uch32)– q;
      sh{ v_n }←(uch32)– mVI n;
      sh{ v_p }←(uch32)– mVI p;
      Ews{ c121665 }←(lg16)– mVI64 c_121665)
(*----------------------------------------*)
POST [ tint ]
PROP (Forall (λx ↦ 0 ≤ x < 2^8) (RFC n p);
      Zlength (RFC n p) = 32)
LOCAL(temp ret_temp (Vint Int.zero))
SEP  (sh{ v_q }←(uch32)– mVI (RFC n p);
      sh{ v_n }←(uch32)– mVI n;
      sh{ v_p }←(uch32)– mVI p;
      Ews{ c121665 }←(lg16)– mVI64 c_121665
```

TweetNaCl implements correctly the RFC

The implementation of X25519 in TweetNaCl (`crypto_scalarmult`) matches the specifications of RFC 7748 (RFC).

More formally:

```
Theorem body_crypto_scalarmult:
  (* VST boiler plate. *)
  semax_body
    (* Clight translation of TweetNaCl. *)
    Vprog
    (* Hoare triples for function calls. *)
    Gprog
    (* function we verify. *)
    f_crypto_scalarmult_curve25519_tweet
    (* Our Hoare triple, see below. *)
    crypto_scalarmult_spec .
```

Formalization of Elliptic Curves

Formal definition of a point

```
Inductive point (𝕂: Type) : Type :=
  (* A point is either at Infinity *)
  | EC_Inf : point 𝕂
  (* or (x, y) *)
  | EC_In : 𝕂 → 𝕂 → point 𝕂.

Notation "∞" := (@EC_Inf _).
Notation "(| x , y |)" := (@EC_In _ x y).



(* Get the x coordinate of p or 0 *)
Definition point_x0 (p : point 𝕂) :=
  if p is (| x, _ |) then x else 0.

Notation "p.x" := (point_x0 p).
```

A Formal Library for Elliptic Curves in the Coq Proof Assistant – Evmorfia-Iro Bartzia, Pierre-Yves Strub https://hal.inria.fr/hal-01102288

Definition

Let $a \in \mathbb{K}\backslash\{-2,2\}$, and $b \in \mathbb{K}\backslash\{0\}$. The elliptic curve $M_{a,b}$ is defined by the equation:

$$by^2 = x^3 + ax^2 + x,$$

$M_{a,b}(\mathbb{K})$ is the set of all points $(x,y) \in \mathbb{K}^2$ satisfying the $M_{a,b}$ along with an additional formal point $\mathcal{O}$, "at infinity".

```
(* B y = x³ + A x² + x *)
Record mcuType := { A: 𝕂; B: 𝕂; _ : B ≠ 0; _ : A² ≠ 4 }

(* is a point p on the curve? *)
Definition oncurve (p : point K) :=
if p is (| x, y |)
  then cB * y² == x³ + cA * x² + x
  else true.

(* We define a point on a curve as a point and the proof that it is on the curve *)
Inductive mc : Type := MC p of oncurve p.
```

Formal definition of the operations over a curve

```
Definition neg (p: point 𝕂) :=
  if p is (| x, y |) then (| x, − y |) else ∞.


Definition add (p1 p2: point 𝕂) :=
    match p1 , p2 with
    | ∞, _ ⇒ p2                                        (* If one point is infinity *)
    | _, ∞ ⇒ p1                                        (* If one point is infinity *)

    | (| x1, y1 |), (| x2, y2 |) ⇒
      if x1 == x2 then
        if (y1 == y2) && (y1 ≠ 0) then ...                     (* If p1 = p2 *)
        else
          ∞
      else                                                     (* If p1 ≠ p2 *)
        let s := (y2 − y1) / (x2 − x1) in
        let xs := s^2 * B − A − x1 − x2 in
         (| xs, −s * (xs − x1) − y1|)
    end


Notation "− x" := (neg x).
Notation "x + y" := (add x y).
Notation "x − y" := (x + (− y)).
```

We define χ and χ_0 to return the x-coordinate of points on a curve.

Definition

Let χ and χ_0:
– $\chi : M_{a,b}(\mathbb{K}) \to \mathbb{K} \cup \{\infty\}$
such that $\chi(\mathcal{O}) = \infty$ and $\chi((x, y)) = x$.
– $\chi_0 : M_{a,b}(\mathbb{K}) \to \mathbb{K}$
such that $\chi_0(\mathcal{O}) = 0$ and $\chi_0((x, y)) = x$.

Montgomery curves make use of projective coordinates. Points are represented with triples $(X : Y : Z)$, with the exception of $(0 : 0 : 0)$

For all $\lambda \neq 0$, the triples $(X : Y : Z)$ and $(\lambda X : \lambda Y : \lambda Z)$ represent the same point.
For $Z \neq 0$, the projective point $(X : Y : Z)$ corresponds to the point $(X/Z, Y/Z)$ on the affine plane.
Likewise the point (X, Y) on the affine plane corresponds to $(X : Y : 1)$ on the projective plane.

Lemma

Let $M_{a,b}$ be a Montgomery curve such that $a^2 - 4$ is not a square in $\mathbb{K}$, and let $X_1, Z_1, X_2, Z_2, X_4, Z_4 \in \mathbb{K}$, such that $(X_1, Z_1) \neq (0,0)$, $(X_2, Z_2) \neq (0,0)$, $X_4 \neq 0$ and $Z_4 \neq 0$. Define

$$X_3 = Z_4((X_1 - Z_1)(X_2 + Z_2) + (X_1 + Z_1)(X_2 - Z_2))^2$$
$$Z_3 = X_4((X_1 - Z_1)(X_2 + Z_2) - (X_1 + Z_1)(X_2 - Z_2))^2,$$

then for any point P_1 and P_2 in $M_{a,b}(\mathbb{K})$ such that $X_1/Z_1 = \chi(P_1), X_2/Z_2 = \chi(P_2)$, and $X_4/Z_4 = \chi(P_1 - P_2)$, we have $X_3/Z_3 = \chi(P_1 + P_2)$.

Remark: *These definitions should be understood in $\mathbb{K} \cup \{\infty\}$. If $x \neq 0$ then we define $x/0 = \infty$.*

Lemma

Let $M_{a,b}$ be a Montgomery curve such that $a^2 - 4$ is not a square in $\mathbb{K}$, and let $X_1, Z_1 \in \mathbb{K}$, such that $(X_1, Z_1) \neq (0, 0)$. Define

$$c = (X_1 + Z_1)^2 - (X_1 - Z_1)^2$$

$$X_3 = (X_1 + Z_1)^2 (X_1 - Z_1)^2$$

$$Z_3 = c\left((X_1 + Z_1)^2 + \frac{a-2}{4} \times c\right),$$

then for any point P_1 in $M_{a,b}(\mathbb{K})$ such that $X_1/Z_1 = \chi(P_1)$, we have $X_3/Z_3 = \chi(2P_1)$.

Correctness of the Montgomery ladder

By combinning the Montgomery ladder with the previous formula, we define a ladder `opt_montgomery` (in which $\mathbb{K}$ has not been fixed yet).

Hypothesis

$a^2 - 4$ is not a square in $\mathbb{K}$.

We prove its correctness.

Theorem

For all $n, m \in \mathbb{N}$, $x \in \mathbb{K}$, $P \in M_{a,b}(\mathbb{K})$, if $\chi_0(P) = x$ then `opt_montgomery` returns $\chi_0(n \cdot P)$

```
Theorem opt_montgomery_ok (n m: nat) (x : K) :
  n < 2^m →
  forall  (p : mc M), p#x0 = x
  (* if x is the x-coordinate of P *)
  → opt_montgomery n m x = (p *+ n)#x0.
  (* opt_montgomery n m xp is the x-coordinate of n · P *).
Qed.
```

Problem !

$a^2 - 4$ **is a square in** $\mathbb{F}_{p^2}$

Curve25519 ladder

```
(* a^2−4 is not a square in F_{2^255−19}. *)
Fact a_not_square : forall  x: F_{2^255−19},
  x^2 ≠ (Zmodp.pi 486662)^2 − 4.

(* 2 is  not a square in F_{2^255−19}. *)
Fact two_not_square : forall  x: F_{2^255−19},
  x^2 ≠ 2 .
```

We now consider $M_{486662,1}(\mathbb{F}_p)$ and $M_{486662,2}(\mathbb{F}_p)$, one of its quadratic twists.

Definition

We instantiate `opt_montgomery` *in two specific ways:*

– $Curve25519\_Fp(n, x)$ *for* $M_{486662,1}(\mathbb{F}_p)$.

– $Twist25519\_Fp(n, x)$ *for* $M_{486662,2}(\mathbb{F}_p)$.

$Curve25519\_Fp(n, x)$ and $Twist25519\_Fp(n, x)$ do not depend on b.

```
Lemma curve_twist_eq: forall  n x,
  curve25519_Fp_ladder n x = twist25519_Fp_ladder n x.
Qed.
```

Correct on $\mathbb{F}_p$

We derive the following two lemmas:

Lemma

For all $x \in \mathbb{F}_p,\ n \in \mathbb{N},\ P \in \mathbb{F}_p \times \mathbb{F}_p$, such that $P \in M_{486662,1}(\mathbb{F}_p)$ and $\chi_0(P) = x$.
Given n and x, $Curve25519\_Fp(n, x) = \chi_0(n \cdot P)$.

Lemma

For all $x \in \mathbb{F}_p,\ n \in \mathbb{N},\ P \in \mathbb{F}_p \times \mathbb{F}_p$ such that $P \in M_{486662,2}(\mathbb{F}_p)$ and $\chi_0(P) = x$.
Given n and x, $Twist25519\_Fp(n, x) = \chi_0(n \cdot P)$.

On the Twist or the Curve over $\mathbb{F}_p$

As 2 is not a square in $\mathbb{F}_p$ we have:

Lemma

For all x in $\mathbb{F}_p$, there exists y in $\mathbb{F}_p$ such that $y^2 = x \ \vee \ 2y^2 = x$

Thus:

Lemma

For all $x \in \mathbb{F}_p$, there exists a point P in $M_{486662,1}(\mathbb{F}_p)$ or in $M_{486662,2}(\mathbb{F}_p)$ such that the x-coordinate of P is x.

And formally:

```
Lemma x_is_on_curve_or_twist :
  forall  x : 𝔽_{2^255 − 19},
  (exists  (p : mc curve25519_mcuType), p#x0 = x) ∨
  (exists  (p' : mc twist25519_mcuType), p'#x0 = x).
Qed.
```

From $\mathbb{F}_p$ to $\mathbb{F}_{p^2}$ and vice-versa

We define the two morphism:

Definition

Define the functions φ_c, φ_t *and* ψ

– $\varphi_c : M_{486662,1}(\mathbb{F}_p) \mapsto M_{486662,1}(\mathbb{F}_{p^2})$ *such that* $\varphi((x,y)) = ((x,0),(y,0))$.

– $\varphi_t : M_{486662,2}(\mathbb{F}_p) \mapsto M_{486662,1}(\mathbb{F}_{p^2})$ *such that* $\varphi((x,y)) = ((x,0),(0,y))$.

– $\psi : \mathbb{F}_{p^2} \mapsto \mathbb{F}_p$ *such that* $\psi(x,y) = (x)$.

And prove:

Lemma

For all $n \in \mathbb{N}$, *for all point* $P \in \mathbb{F}_p \times \mathbb{F}_p$ *on the curve* $M_{486662,1}(\mathbb{F}_p)$ *(respectively on the quadratic twist* $M_{486662,2}(\mathbb{F}_p)$*), we have:*

$$P \in M_{486662,1}(\mathbb{F}_p) \implies \varphi_c(n \cdot P) = n \cdot \varphi_c(P)$$
$$P \in M_{486662,2}(\mathbb{F}_p) \implies \varphi_t(n \cdot P) = n \cdot \varphi_t(P)$$

Notice that:

$$\forall P \in M_{486662,1}(\mathbb{F}_p), \quad \psi(\chi_0(\varphi_c(P))) = \chi_0(P)$$
$$\forall P \in M_{486662,2}(\mathbb{F}_p), \quad \psi(\chi_0(\varphi_t(P))) = \chi_0(P)$$

Theorem

For all $n \in \mathbb{N}$, such that $n < 2^{255}$, for all $x \in \mathbb{F}_p$ and $P \in M_{486662,1}(\mathbb{F}_{p^2})$ such that $\chi_0(P) = x$, $Curve25519\_Fp(n, x)$ computes $\chi_0(n \cdot P)$.

which is formalized in Coq as:

```
Theorem curve25519_Fp2_ladder_ok:
  forall  (n : nat) (x: 𝔽_{2^255 − 19}),
  (n < 2^255)%nat →
  forall  (p  : mc curve25519_Fp2_mcuType),
  p #x0 = Zmodp2.Zmodp2 x 0 →
  curve25519_Fp_ladder n x = (p *+ n)#x0 /p.
Qed.
```

The implementation of X25519 in TweetNaCl computes the $\mathbb{F}_p$-restricted x-coordinate scalar multiplication on $E(\mathbb{F}_{p^2})$ where p is $2^{255} - 19$ and E is the elliptic curve $y^2 = x^3 + 486662x^2 + x$.

```
Theorem RFC_Correct: forall (n p : list Z)
  (P:mc curve25519_Fp2_mcuType),
  Zlength n = 32 →
  Zlength p = 32 →
  Forall (λ x ⇒ 0 ≤ x ∧ x < 2 ^ 8) n →
  Forall (λ x ⇒ 0 ≤ x ∧ x < 2 ^ 8) p →
  Fp2_x (decodeUCoordinate p) = P#x0 →
  RFC n p =
    encodeUCoordinate
      ((P *+ (Z.to_nat (decodeScalar25519 n))) _x0).
Qed.
```

Thank you.

Equivalences

Generic Operations

```
Class Ops (T T': Type) (Mod: T → T):=
{
  A:   T → T → T;                  (* Addition       over T *)
  M:   T → T → T;                  (* Multiplication over T *)
  Zub: T → T → T;                  (* Subtraction    over T *)
  Sq:  T → T;                      (* Squaring       over T *)
  C_0: T;                          (* Constant 0       in T *)
  C_1: T;                          (* Constant 1       in T *)
  C_121665: T;                     (* Constant 121665  in T *)
  Sel25519: ℤ → T → T → T;         (* Select the 2nd or 3rd argument depending of Z *)
  Getbit:   ℤ → T' → ℤ;            (* Return the ith bit of T' *)

  (* Mod conservation *)
  Mod_ZSel25519_eq : forall b p q,  Mod (Sel25519 b p q) = Sel25519 b (Mod p) (Mod q);
  Mod_ZA_eq :        forall p q,    Mod (A p q)          = Mod (A (Mod p) (Mod q));
  Mod_ZM_eq :        forall p q,    Mod (M p q)          = Mod (M (Mod p) (Mod q));
  Mod_ZZub_eq :      forall p q,    Mod (Zub p q)        = Mod (Zub (Mod p) (Mod q));
  Mod_ZSq_eq :       forall p,      Mod (Sq p)           = Mod (Sq (Mod p));

  Mod_red :          forall p,      Mod (Mod p)          = (Mod p)
}.
```

Generic Montgomery Ladder

```
Context {T : Type}.
Context {T' : Type}.
Context {Mod : T → T}.
Context {O : Ops T T' Mod}.

Fixpoint montgomery_rec (m : ℕ) (z : T') (a b c d e f x : T) : (T * T * T * T * T * T) :=
  match m with
  | 0 ⇒ (a,b,c,d,e,f)
  | S n ⇒
      let r := Getbit (ℤ.of_nat n) z in
      let (a, b) := (Sel25519 r a b, Sel25519 r b a) in
      let (c, d) := (Sel25519 r c d, Sel25519 r d c) in
      let e := A a c in
      let a := Zub a c in
      let c := A b d in
      let b := Zub b d in
      let d := Sq e in
      let f := Sq a in
      let a := M c a in
      let c := M b e in
      let e := A a c in
      let a := Zub a c in
      let b := Sq a in
      let c := Zub d f in
      let a := M c C_121665 in
      let a := A a d in
      let c := M c a in
      let a := M d f in
      let d := M b x in
      let b := Sq e in
      let (a, b) := (Sel25519 r a b, Sel25519 r b a) in
      let (c, d) := (Sel25519 r c d, Sel25519 r d c) in
      montgomery_rec n z a b c d e f x
    end.
```

Operations Equivalence

```
Class Ops_Mod_P {T T' U:Type}
                {Mod:U → U} {ModT:T → T}
                `(Ops T T' ModT) `(Ops U U Mod)  :=
{
P:  T  →  U;     (* Projection from T  to U *)
P': T' →  U;     (* Projection from T' to U *)

A_eq:        forall a b, Mod (P (A a b))   = Mod (A (P a) (P b));
M_eq:        forall a b, Mod (P (M a b))   = Mod (M (P a) (P b));
Zub_eq:      forall a b, Mod (P (Zub a b)) = Mod (Zub (P a) (P b));
Sq_eq:       forall a,   Mod (P (Sq a))    = Mod (Sq (P a));

C_121665_eq: P C_121665 = C_121665;
C_0_eq:      P C_0 = C_0;
C_1_eq:      P C_1 = C_1;

Sel25519_eq: forall b p q,  Mod (P (Sel25519 b p q)) = Mod (Sel25519 b (P p) (P q));
Getbit_eq:   forall i p,    Getbit i p = Getbit i (P' p);
}.
```

Generic Montgomery Equivalence

```
Context {T:    Type}.
Context {T':   Type}.
Context {U:    Type}.
Context {ModT: T → T}.
Context {Mod:  U → U}.
Context {TO:   Ops T T' ModT}.
Context {UO:   Ops U U  Mod}.
Context {UTO: @Ops_Mod_P T T' U Mod ModT TO UO}.


(* montgomery_rec over T is equivalent to montgomery_rec over U *)
Corollary montgomery_rec_eq_a: forall (n:ℕ) (z:T') (a b c d e f x: T),
  Mod (P (get_a (montgomery_rec n z a b c d e f x))) =                  (* over T *)
  Mod (get_a (montgomery_rec n (P' z) (P a) (P b) (P c) (P d) (P e) (P f) (P x))).  (* over U *)
Qed.

Corollary montgomery_rec_eq_c: forall (n:ℕ) (z:T') (a b c d e f x: T),
  Mod (P (get_c (montgomery_rec n z a b c d e f x))) =                  (* over T *)
  Mod (get_c (montgomery_rec n (P' z) (P a) (P b) (P c) (P d) (P e) (P f) (P x))).  (* over U *)
Qed.
```

Instanciating

```
Definition modP (x: ℤ) := x mod 2^255 − 19.

(* Operations over ℤ *)
Instance Z_Ops :      Ops ℤ ℤ modP := {}.

(* Operations over 𝔽_{2^255 − 19} *)
Instance Z25519_Ops : Ops 𝔽_{2^255 − 19} ℕ id := {}.

(* Equivalence between ℤ (with modP) and ℤ *)
Instance Zmod_Z_Eq :   @Ops_Mod_P Z Z Z modP modP Z_Ops Z_Ops :=
{ P := modP; P' := id }.

(* Equivalence between ℤ (with modP) and 𝔽_{2^255 − 19} *)
Instance Z25519_Z_Eq : @Ops_Mod_P 𝔽_{2^255 − 19} nat Z modP id Z25519_Ops Z_Ops :=
{ P := val; P' := ℤ.of_nat }.

Inductive List16 (T:Type) := Len (l:list T): Zlength l = 16 → List16 T.
Inductive List32B :=         L32B (l:list ℤ): Forall (λx ⇒ 0 ≤ x < 2^8) l → List32B.

(* Operations over List16,List32 *)
Instance List16_Ops : Ops (@List16 ℤ) List32B id := {}.

(* Equivalence between List16,List32 and ℤ *)
Instance List16_Z_Eq : @Ops_Mod_P (@List16 ℤ) (List32B) Z modP id List16_Ops Z_Ops :=
{  P l := (ZofList 16 (List16_to_List l));       P' l := (ZofList 8 (List32_to_List l));  }.

(* Operations over list of ℤ *)
Instance List_Z_Ops : Ops (list ℤ) (list ℤ) id := {}.

(* Equivalence between List16,List32 and list of ℤ *)
Instance List16_List_Z_Eq : @Ops_Mod_P (List16 ℤ) (List32B) (list ℤ) id id List16_Ops List_Z_Ops :=
{  P := List16_to_List;         P' := List32_to_List      }.
```

Full Equivalence